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ON A METHOD OF MINIMIZING THE MAXIMAL ACCUMULATED ERROR

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ON A METHOD OF MINIMIZING THE MAXIMAL ACCUMULATED ERROR

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1. Let us consider an automatic control system described by the differential equation /74*

$$\begin{aligned} &L[y(t)] = f(t), \quad 0 \leqslant t \leqslant T, \\ &y(0) = \dot{y}(0) = \ldots = y^{(n-1)}(0) = 0, \end{aligned} \tag{1}$$

where

$$\begin{aligned} &L[y(t)] = L_1[y(t)] + c(t)\,y(t), \\ &L_1 = \frac{d^n}{dt^n} + a_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_0(t). \end{aligned}$$

We shall assume that the function $a_k(t)$ at $t \in [0, T]$ has k continuous derivatives $(k = 0, 1, \ldots, n-1)$, and

$$\max|a_k(t)| = a, \qquad 0 \leqslant t \leqslant T, \quad 0 \leqslant k \leqslant n-1;$$

The quantity c(t) belongs to the class H of piecewise continuous functions and $|c(t)| \leqslant c_0, 0 \leqslant t \leqslant T$. The function c(t) has the physical meaning of a variable amplification factor (or variable rigidity in the case of mass oscillations on a spring). The perturbation f(t) belongs to the class F of piecewise continuous functions and $|f(t)| \leqslant f_0, \ 0 \leqslant t \leqslant T$. It is necessary to determine the function $c^0(t) \in H$, for which the following is satisfied

$$A = \min_{c \in H} \max_{f \in F} |y(T, f(t), c(t))|. \tag{2}$$

Here y[T, f(t), c(t)] is the solution of (1) at the moment of time T for the selected functions c(t) and f(t). This problem

*Numbers in the margin indicate pagination of foreign text.

was formulated in [1] and [2]. These studies give the algorithm for the successive minimization of the maximum accumulated error. The sufficient conditions are given, for which the algorithm leads to a definition of A in (2). An estimate is made of the number of conversions for the function $c^\circ(t)$ and the number of steps necessary to obtain A.

This article describes another simpler algorithm for minimizing the maximum accumulated error. A detailed study is made of the oscillatory system of the second order /75

$$\left(L_1 = \frac{d^2}{dt^2} + k\right),$$

for which when $T > \pi(k+c_0)^{-\frac{1}{2}}$ it is shown that A is reached for the function $c^\circ(t)$, which differs from a constant (for the mechanical interpretation, this means that the spring must have a rigidity which is variable in time). In one particular case, it is shown that this algorithm leads to a definition of the global minimum. An example is given of a system for which the algorithm was realized on a computer.

2. Let us describe a method for the successive minimization of the maximum accumulated error. Let us assume that the function $c_i(t)$ is determined on the i^{th} step. The iteration process consists of establishing $c_{i+1}(t)$, according to the value of $c_i(t)$ obtained, which gives the minimized maximum error. Let us introduce the notation: $G_i(t) = G_i(t, T)$ — the Cauchy function of the equation $L_1(y) + c_i(t)y = 0$; $f_i(t) = f_0 \operatorname{sgn} G_i(t)$; $\theta_i(t) = |c_i(t) - c_0 \operatorname{sgn}[G_i(t) y_i(t)]|$; $0 \leqslant t \leqslant T$, where $y_i(t)$ is the solution of (1) when $c(t) = c_i(t)$, $f(t) = f_i(t)$. We should note that writing the solution of (1) when $c(t) = c_i(t)$ in the form

$$y(T) = \int_0^T G_i(t) f(t)\, dt$$

leads to the fact that $\max_{f \in F}|y(T)|$ is achieved for the function $f(t)=f_i(t),\ 0 \leqslant t \leqslant T$. Let us assume that B is any upper boundary of the function $|G_i(\tau, t)|$ when $\tau \in [0, t],\ t \in [0, T],\ c_i(t) \in H$, — for example (see [3]):

$$B=\exp[T\sqrt{1+(a+c_0)^2}],$$

$$D_i=\int_0^T \theta_i(t)\cdot|G_i(t)\,y_i(t)|\,dt \Big/ \int_0^T \theta_i(t)\,dt,$$

$$\alpha_i=D_i/4B^2\int_0^T \theta_i(t)\cdot|y_i(t)|\,dt.$$

Let us determine the set $m_i(\lambda)$:

$$m_i(\lambda)=\left\{t: t\in[0,\ T],\quad |G_i(t)|\leqslant B^2\lambda\int_0^T \theta_i(t)\,dt\right\},\quad \lambda \geqslant 0.$$

Let us introduce the number β_i as follows: if $0<D_i \leqslant 16B^2Tf_0$, then β_i is the root of the equation

$$\operatorname{mes}[m_i(\lambda)]=D_i/16B^2f_0; \tag{3}$$

If $D_i>16B^2Tf_0$, then β_i is any positive number for which $m_i(\beta_i)=[0,\ T]$. It may be shown that when $0<D_i \leqslant 16B^2Tf_0$, Equation (3) has a unique non-zero solution, and $\beta_i=0$ only when $D_i=0$. We should note that if β_i is a discontinuity point of the function $\operatorname{mes}[m_i(\lambda)]$ and $\operatorname{mes}[m_i(\beta_i+0)]>D_i/16B^2f_0$, but $\operatorname{mes}[m_i(\beta_i-0)]<D_i/16B^2f_0$, we then assume

$$\operatorname{mes}[m_i(\beta_i)]=D_i/16B^2f_0.$$

The equation $c_{i+1}(t)$ gives the minimized maximum accumulated error and is determined by the formula

$$l_i(t)=c_{i+1}(t)-c_i(t)=\gamma_i\theta_i(t)\operatorname{sgn}[G_i(t)\,y_i(t)],\qquad 0 \leqslant t \leqslant T, \tag{4}$$

where $\gamma_i=\min\{\alpha_i,\ \beta_i\}$. Here the numbers α_i and β_i are the same as were determined above.

3. In this section, we shall determine the minimization /76
of the maximum accumulated error, if $c_{i+1}(t)$ is determined by Formula (4).

We shall first prove that when $0 \leqslant t_0 \leqslant T$ the following inequality holds

$$|G_i(t_0) - G_{i+1}(t_0)| \leqslant B^2 \int_0^T |l_i(t)|\, dt. \qquad (5)$$

Actually, we shall set

$$h(t) = G_i(t_0,\ t) - G_{i+1}(t_0,\ t), \qquad 0 \leqslant t_0 \leqslant t \leqslant T.$$

It follows from the definition of the Cauchy function $G_i(t)$ and $G_{i+1}(t)$ that h(t) satisfies the equation

$$L_1[h(t)] + c_i(t)\, h(t) = [c_{i+1}(t) - c_i(t)]\, G_{i+1}(t_0,\ t).$$

In addition, taking the fact into account that the initial conditions for the function h(t) are zero, we obtain

$$h(T) = \int_{t_0}^{T} G_i(t,\ T)\, l_i(t)\, G_{i+1}(t_0,\ t)\, dt.$$

Inequality (5) thus follows. We obtain the following from (5) and the definition of the set $m_i(\gamma_i)$

$$f_i(t) - f_{i+1}(t) = 0 \text{ for } t \in [0,\ T] \setminus m_i(\gamma_i). \qquad (6)$$

Let us write the solution $y_i(t)$ as follows

$$y_i(T) = y_{i+1}(T) + \int_0^T l_i(t) G_i(t)\, y_i(t)\, dt + \int_0^T G_{i+1}(t) \cdot [f_i(t) - f_{i+1}(t)]\, dt + \\ + \int_0^T [G_i(t) - G_{i+1}(t)]\, l_i(t)\, y_i(t)\, dt. \qquad (7)$$

Taking (4), (5), and (6) into account, we obtain

$$\int_0^T l_i(t) G_i(t) y_i(t)\, dt = \gamma_i \int_0^T \theta_i(t) \cdot |G_i(t) y_i(t)|\, dt,$$

$$\left|\int_0^T G_{i+1}(t) \cdot [f_i(t) - f_{i+1}(t)]\, dt\right| \leqslant 2f_0 \int_{t \in m_i(\gamma_i)} |G_{i+1}(t)|\, dt \leqslant$$

$$\leqslant 2f_0 2B^2 \gamma_i \int_0^T \theta_i(t)\, dt\, \mathrm{mes}\,[m_i(\beta_i)] \leqslant \frac{\gamma_i}{4} \int_0^T \theta_i(t) \cdot |G_i(t) y_i(t)|\, dt,$$

$$\left|\int_0^T [G_{i+1}(t) - G_i(t)]\, l_i(t) y_i(t)\, dt\right| \leqslant B^2 \int_0^T |l_i(t)|\, dt \int_0^T |l_i(t) y_i(t)|\, dt \leqslant$$

$$\leqslant B^2 \gamma_i \int_0^T \theta_i(t)\, dt\, \alpha_i \int_0^T \theta_i(t) \cdot |y_i(t)|\, dt = \frac{\gamma_i}{4} \int_0^T \theta_i(t) |G_i(t) y_i(t)|\, dt.$$

It follows from these determinations that /77

$$\int_0^T G_{i+1}(t) [f_i(t) - f_{i+1}(t)]\, dt \geqslant -\frac{\gamma_i}{4} \int_0^T \theta_i(t) \cdot |G_i(t) y_i(t)|\, dt$$

$$\int_0^T [G_{i+1}(t) - G_i(t)] \cdot l_i(t) \cdot y_i(t)\, dt \geqslant -\frac{\gamma_i}{4} \int_0^T \theta_i(t) \cdot |G_i(t) y_i(t)|\, dt$$

Substituting these relationships in (7), we obtain

$$y_{i+1}(T) \leqslant y_i(T) - \frac{\gamma_i}{2} \int_0^T \theta_i(t) \cdot |G_i(t) y_i(t)|\, dt.$$

It thus follows that a necessary condition for the optimality of controlling $c_i(t)$ is the equation $\theta_i(t) = 0,\ 0 \leqslant t \leqslant T$, or

$$c_i(t) = c_0 \,\mathrm{sgn}\,[G_i(t) \cdot y_i(t)] \tag{8}$$

for almost all $t \in [0, T]$.

For an unequivocal interpretation of (8), it must be shown that the set of zeros of the function $G_i(t) y_i(t)$ has the dimension of zero.

In the same way, this set is finite. Actually, for the function $G_1(t)$ this was proven according to a method proposed in [4]. It may then be found in a similar manner that the set of zeros of the function $y_1(t)$ is finite.

4. Let us consider the problem for the operator

$$L = \frac{d^2}{dt^2} + [k + c(t)],$$

where $k = \text{const}, \; k - c_0 > 0$. The following three cases are studied here:

a) $0 \leqslant T \leqslant \pi (k + c_0)^{-\frac{1}{2}}$;

b) $\pi (k + c_0)^{-\frac{1}{2}} < T < \infty$;

c) $\pi (k + c_0)^{-\frac{1}{2}} < T \leqslant 2\pi (k + c_0)^{-\frac{1}{2}}, \quad c_0 < k \leqslant (5/3) c_0.$

Let us examine each case separately.

a) It follows from the Sturm theorem regarding the number of zeros of the solution for the equation $L(y) = 0$ (see [3]) that $G_0(t)$ does not have zeros in the $(0, T)$ interval for any $c(t) \in H$. Taking the fact into account that in this case

$$y_0(t) = f_0 \int_0^t G_0(\tau, t)\, d\tau,$$

we find that $y_0(t)$ does not have zeros either in $(0, T)$. Based on (8), we conclude that the constant c_0 is a single optimal control in this case.

b) In this case $c(t)=a=\mathrm{const}$ $(0\leqslant t\leqslant T, -c_0\leqslant a\leqslant c_0)$ is not optimal for any a. Actually, we shall consider one iteration of the algorithm in Section 2. For the first step, let us select

$$c_1(t) = a = \mathrm{const}, \qquad 0 \leqslant t \leqslant T;$$

Then,

$$G_1(t) = G_1(t,\ T) = (k+a)^{-\frac{1}{2}} \sin[(k+a)^{\frac{1}{2}}(T-t)]. \tag{9}$$

Let us assume t_1 is the first positive zero of the function $G_1(t)$. Then /78

$$y_1(t) = f_0\cdot(k+a)\cdot[\operatorname{sgn} G_1(t)]\cdot[1-\cos(t(k+a)^{\frac{1}{2}})], \qquad 0\leqslant t\leqslant t_1). \tag{10}$$

We find from (9) that

$$t_1 \leqslant \pi(k+a)^{-\frac{1}{2}},$$

and from (10), consequently,

$$y_1(t) \neq 0 \text{ for } 0 < t \leqslant t_1.$$

Thus, there is a $t_2 > t_1$ such that

$$y_1(t)\cdot G_1(t) < 0 \text{ for } t_1 < t < t_2.$$

Thus, if $0\leqslant a\leqslant c_0$, we take $l_1(t)\neq 0$ for (t_1, t_2) ; if $c_0\leqslant a<0$ then $l_1(t)\neq 0$ for $[0, t_1]$. Then the new control $c_2(t)$ will be different from the constant a for (t_1, t_2) , because for $[0, t_1]$ the maximum accumulated error $y_2(T)$ is realized which is less than $y_1(T)$.

c) Let us assume G(t, T) is the Cauchy function for the equation

$$\ddot{y} + [k + c(t)]\,y = 0. \tag{11}$$

where $p(t)$ is the solution of the equation (11) under the conditions $p(T) = 0$, $\dot{p}(T) = 1$. It is known (see [3]) that $G(t, T) = -p(t)$, $0 \leqslant t \leqslant T$. Then Condition (8) has the form

$$c^\circ(t) = -c_0 \operatorname{sgn}[p^\circ(t) \cdot y^\circ(t)], \qquad 0 \leqslant t \leqslant T. \tag{12}$$

Since

$$T > \pi (k + c_0)^{-\frac{1}{2}},$$

$p^\circ(t)$ has at least one zero for $(0, T)$. It follows from

$$T \leqslant 2\pi (k + c_0)^{-\frac{1}{2}}$$

and the Shturm theorem that $p^\circ(t)$ has only one zero. Let us assume $p^\circ(\alpha) = 0$; then

$$p^\circ(t) \geqslant 0 \text{ for } 0 \leqslant t \leqslant \alpha; \quad p^\circ(t) < 0 \text{ for } \alpha < t \leqslant T.$$

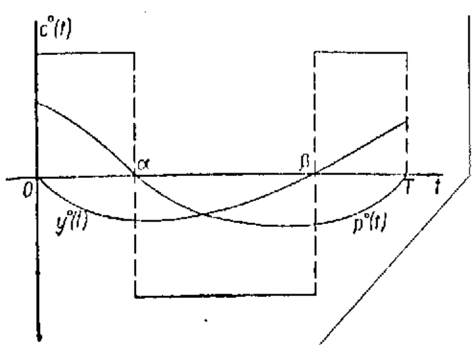
The maximizing perturbation is: $f^\circ(t) \leqslant 0$ when $0 \leqslant t \leqslant \alpha$; $f^\circ(t) > 0$ for $\alpha < t \leqslant T$. Therefore, $y^\circ(t) < 0$ for $0 \leqslant t \leqslant \alpha$. However,

$$y^\circ(T) = f_0 \int_0^T |G_0(t, T)| \, dt > 0.$$

Thus, $y^\circ(t)$ has at least one zero for $(0, T)$. Utilizing the inequality

$$T \leqslant 2\pi (k + c_0)^{-\frac{1}{2}}$$

we find that $y^\circ(t)$ has only one zero. Let us set $y^\circ(\beta) = 0$. We thus find from (12) that the optimal control has the form shown in the figure.



Let us now prove the uniqueness of this control, i.e., the sufficiency of the equation (12). We shall assume the

existence of the functions $\overline{p_1}(t), y_1(t)$ and $c_1(t)$ which satisfy the condition (12) and differ from $p^0(t), y^0(t)$, and $c^0(t)$. Let us assume $y_1(\beta_1)=0, \ p_1(\alpha_1)=0$. Using the Shturm theorom, we find either $[\alpha_1, \beta_1]\subset$ or $\subset[\alpha, \beta]$. Let us assume that $[\alpha_1, \beta_1]\subset[\alpha, \beta]$. /79
Then

$$c_1(t) \geqslant c^\circ(t), \quad f_1(t) \leqslant f^\circ(t), \qquad 0 \leqslant t \leqslant T.$$

We have

$$\ddot{y}_1 + (k + c_1(t))\, y_1 = f_1(t),$$
$$\ddot{y}^\circ + (k + c^\circ(t))\, y^\circ = f^\circ(t).$$

Subtracting the first equation from the second, we obtain

$$\frac{d^2}{dt^2}(y^\circ - y_1) + (k + c^\circ(t))\cdot(y^\circ - y_1) = (f^\circ(t) - f_1(t)) + (c_1(t) - c^\circ(t))\, y_1.$$

Integrating the latter equation over the segment $[0, \beta_1]$:

$$y^\circ(t) - y_1(t) = \int_0^t G_0(\tau, t)\cdot[(f^\circ(\tau) - f_1(\tau)) + (c_1(\tau) - c^\circ(\tau))\, y_1(\tau)]\, d\tau =$$

$$= \int_\alpha^t G_0(\tau, t)\cdot[(f^\circ(\tau) - f_1(\tau)) + (c_1(\tau) - c^\circ(\tau))\, y_1(\tau)]\, d\tau.$$

When $t = \beta_1$, we obtain

$$y^\circ(\beta_1) - y_1(\beta_1) = \int_\alpha^{\alpha_1} G_0(\tau, \beta_1)\cdot(2f_0 + 2c_0 y_1(\tau))\, d\tau, \tag{13}$$

since $f^\circ(\tau) = f_1(\tau), \quad c^\circ(\tau) = c_1(\tau); \quad \alpha_1 \leqslant \tau \leqslant \beta_1.$

The function $y_1(\tau)$ when $\alpha \leqslant \tau \leqslant \alpha_1$ has the form

$$y_1(\tau) = -(f_0/(k + c_0))\cdot[1 - \cos(\tau(k + c_0)^{-\frac{1}{2}})].$$

Thus, since $k > c_0$, we obtain

$$|y_1(t)| \leqslant 2f_0/(k + c_0) \leqslant f_0/c_0.$$

Thus,

$$2f_0 + 2c_0 y_1(t) > 0 \text{ for } \alpha \leqslant t \leqslant \alpha_1.$$

Taking the fact into account that when $k < (5/3)c_0$,

$$T \leqslant 2\pi(k + c_0)^{-\frac{1}{2}} < \pi(k - c_0)^{-\frac{1}{2}}$$

will hold, we obtain $G_0(t, \beta_1) > 0, \alpha \leqslant t \leqslant \alpha_1$. Consequently, the integral in (13) is positive; therefore, $y^\circ(\beta_1) - y_1(\beta_1) = y^\circ(\beta_1) > 0.$ The latter inequality contradicts the assumption $\alpha \leqslant \alpha_1 < \beta_1 < \beta$. The case $\alpha_1 < \alpha < \beta < \beta_1$ may be examined in a similar manner. Thus, we obtain the equation $\alpha = \alpha_1, \beta = \beta_1$. In case c) Equation (8) is necessary and sufficient for the optimality of the control $c^\circ(t)$ in the sense of the minimum of the maximum accumulated error.

5. The numerical realization of the algorithm was performed on a "Minsk-22" computer for a system of the second order like (11) with the following values of the parameters $T = \pi = 3{,}14$; $k = 3$; $c_0 = 1$. The results obtained are given below for the six iterations:

1) $y_1(T)$ = 1.0 (constant rigidity of the spring); 2) $y_2(T)$ = 0.91 (variable rigidity); 3) $y_3(T) = 0{,}908$; 4) $y_4(T) = 0{,}9069227$; 5) $y_5(T) =$ $0{,}9037$; 6) $y_6(T) = 0{,}9019117$

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